SDS321 Practice Problems for Exam 2: Solutions

1. Alex is new to basketball. He is learning to shoot the ball through the hoop from the free throw line, which he is currently able to do 20% of the time. He goes to the gym with the intention of practicing just his free throws.

   a) Suppose he is unable to get the ball through the hoop in his first 12 attempts. What is the probability that he will be able to make it for the first time, on his 15th attempt?

      X = number of attempts until first successful shot ~ Geo(0.2)
      $P(X=15 \mid X>12) = P(X=12+3 \mid X>12) = P(X=3) = 0.8 \times 0.8 \times 0.2 = 0.128$

   b) Suppose he is unable to get the ball through the hoop in his first 12 attempts. What is the probability that he will need at least 15 attempts to make his first successful shot?

      $P(X \geq 15 \mid X > 12) = P(X > 14 \mid X > 12) = P(X > (12+2) \mid X>12) = P(X > 2)$
      $= 1 - P(X=1) - P(X=2)$
      $= 1 - 0.2 - (0.8 \times 0.2)$
      $= 1 - 0.2 - 0.16 = 0.64$

2. Suppose 20% of people in a city are estimated to have a disease. You test 20 individuals chosen at random to see if they have the disease. What is the probability that at least two of the people test positive for the disease?

   X = # test +ive ~ Bin(20, 0.2)
   $P(X \geq 2) = 1 - P(X=0) - P(X=1) = 1 - (0.8)^{20} - [20 \times (0.2) \times (0.8)^{19}]$

3. The number of typos in textbooks printed by publishing company XYZ follows a Poisson distribution. It is known that the mean number of typos in 100 pages of their textbooks is 3.2. What is the probability of exactly 10 typos in a 200-page textbook?

   Mean number of typos in 200 pages = 3.2*2 = 6.4

   Ans = $(6.4)^{10} * \exp(-6.4)/10! = 0.053$

4. Assume that X and Y are are two independent random variables, with E[X]=1, E[Y]=3, $E[X^2]=5$, $E[Y^2]=34$. Find

   a) $Var(X) = E[X^2] - (E[X])^2 = 4$

   b) Var(Y)=25

   c) E[5X-Y+100]=5(1)-3+100=102

   d) Var(5X-Y+100)=25(4)+25=125

   e) $E[(Y-2)^2] = E[Y^2] - 4E[Y] + 4 = 34 - 4(3) + 4 = 26$

5. Suppose the marginal distribution of X is as below. Find the expectation and variance of X.

x	1	2	3
$P_X(x)$	2/10	3/10	5/10

E[X] = (1*0.2) + (2*0.3) + (3*0.5) = 0.2+0.6+1.5 = 2.3

$E[X^2]$ = (1*0.2) + (4*0.3) + (9*0.5) = 0.2+1.2+4.5 = 5.9

Var(X) = $E[X^2] - (E[X])^2 = 5.9 - 2.3^2 = 0.61$

6. If X is a Bernoulli random variable with p=0.2, what is $E[10X^3-1]$?

   $E[10X^3-1] = 10*E[X^3] -1 = 10*0.2 - 1= 1$

7. Let X and Y be iid Binomial distributions with n=10 and p=0.6. Let Z = X +Y.

   a) Find the mean and variance of Z.
   E[X] = E[Y] = 10*0.6 = 6, and var(X) = var(Y) = 10*0.6*0.4 = 2.4

   So E[Z] = 12 Var[Z] = 4.8

   b) Find $p_Z(1)$
   = P(X=0 and Y=1) + P(X=1 and Y=0) = $20(0.6)(0.4)^{19}$

8. Charles claims that he can distinguish between whisky and brandy 70% of the time. Let p= Charles' probability of distinguishing between the drinks. Charles' claim is p = p_C = 0.75. Ruth bets that he cannot tell the difference, and in fact, just guesses. So Ruth's claim is p = p_R = 0.5. To settle this, a bet is made. Charles is given 5 glasses, each filled with whisky or brandy, chosen by tossing a fair coin. He wins the bet if he gets 4 or more correct.

   a) If Charles is correct, what is the probability that Charles wins the bet?
   b) If Ruth is correct, what is the probability that Charles wins the bet?
   c) Assume that, before seeing the outcome of the bet, you believe that Charles is right with probability 0.1, and Ruth is right with probability 0.9. You find out that Charles won the bet. With what probability should you believe Charles is right?

(a) If Charles is correct—i.e. if $p = p_C = 0.75$—what is the probability that Charles wins the bet. The question is for $\mathbf{P}(X \geq 4)$ when $p = p_C$. In that case $X \sim \mathit{Binomial}(n, p_C)$ and $\mathbf{P}(X \geq 4) = p_X(4) + p_X(5) = 5p_C^4(1-p_C) + p_C^5 = p_C^4(5 - 4p_C) = 0.63$

(b) If Ruth is correct—i.e. if Charles is guessing randomly—what is the probability that Charles wins the bet.
The question is for $\mathbf{P}(X \geq 4)$ when $p = p_R$. Now $X \sim \mathit{Binomial}(n, p_R)$ and $\mathbf{P}(X \geq 4) = p_X(4) + p_X(5) = 5p_R^4(1-p_R) + p_R^5 = 6\left(\frac{1}{2}\right)^5 = 0.19$

(c) Assume that, before seeing the outcome of the bet, you believe that Charles is right with probability 0.1, and Ruth is right with probability 0.9
I find out that Charles won the bet. With what probability should I believe Charles is right?
Let A be "Charles is correct" and B be "Charles wins the bet". $P(A) = 0.1$, $P(B|A) = 0.63$, $P(B|A^c) = 0.19$, so

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(B|A)\mathbf{P}(A)}{\mathbf{P}(B|A)\mathbf{P}(A) + \mathbf{P}(B|A^c)\mathbf{P}(A^c)} = \frac{.1 \cdot .63}{.1 \cdot .63 + .9 \cdot .19} = 0.27$$

9. On average three traffic accidents occur in a given city per day. You may assume that the number of accidents follows a Poisson distribution with λ=3.

a) That is the probability that we see at least three accidents in a day?
b) If you know there is at least one accident, what is the probability that the total number of accidents is at least three?

(a) What is the probability that we see at least three accidents in a day? $\mathbf{P}(X \geq 3) = 1 - \mathbf{P}(X = 0) - \mathbf{P}(X = 1) - \mathbf{P}(X = 2) = 1 - e^{-3}(1 + 3 + 3^2/2) = 0.577$
(b) If you know there is at least one accident, what is the probability that the total number of accidents is at least three? $\mathbf{P}(X \geq 1) = 1 - \mathbf{P}(X = 0) = 1 - e^{-3} = 0.950$. $\mathbf{P}(X \geq 3|X \geq 1) = \mathbf{P}(X \geq 3)/\mathbf{P}(X \geq 1) = 0.577/0.950 = 0.607$

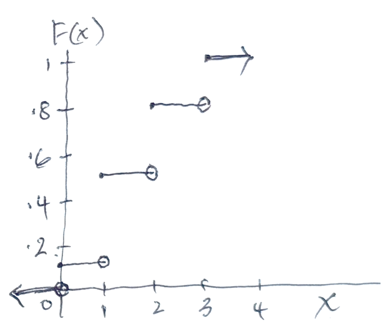
10. Suppose X is a random variable with the following PMF:

$$P_X(k) = \begin{cases} 0.1 & \text{for } k = 0 \\ 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

a) Find and plot the CDF of X.
b) Use the CDF to find the PMF and confirm it is the same as is given.
c) Find E[X]
d) Find var(X)
e) If $Y=(X-2)^2$, find E[Y] and var(Y)

a) CDF of X is:

when $x<0$ $F(x) = 0$
" $0 \leq x < 1$ $F(x) = 0.1$
" $1 \leq x < 2$ $F(x) = 0.5$
" $2 \leq x < 3$ $F(x) = 0.8$
" $x \geq 3$ $F(x) = 1$



b) You can check for yourself.

c) $E[X] = 0(0.1) + 1(0.4) + 2(0.3) + 3(0.2) = 1.6$

d) $E[X^2] = 0^2(0.1) + 1^2(0.4) + 2^2(0.3) + 3^2(0.2) = 0.4+1.2+1.8 = 3.4$
$Var(X) = 3.4 - (1.6)^2 = 0.84$

e) Expectation and variance of Y:

$$E[Y] = E[(X-2)^2] = \sum_x^{t} (x-2)^2 P(X=x)$$

$$= (0-2)^2 * 0.1 + (1-2)^2 * 0.4 + (2-2)^2 * 0.3 + (3-2)^2 * 0.2$$

$$= 0.4 + 0.4 + 0 + 0.2 = 1$$

$$E[Y^2] = E[(X-2)^4]$$

$$= (16 * 0.1) + (1 * 0.4) + 0 + (1 * 0.2)$$

$$= 1.6 + 0.4 + 0.2 = 2.2$$

$$var(Y) = 2.2 - 1^2 = 1.2$$